Completely Additive Linear Forms on von Neumann Algebras

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Abstract

By Jordan decomposition, we shall see that a bounded completely additive linear form on a von Neumann algebra is σ-weakly continuous.

By Jordan decomposition and the concept of singularity, we shall show that a bounded completely additive linear form on a von Neumann algebra is σ-weakly continuous. The proof is distant from the Dixmier's that and extraordinarily simple. Also, without using the concept of singularity and based on only Jordan decomposition, we can prove that.

Lemma 1. *Let A be a C^*-algebra and φ a bounded self-adjoint linear form on A. Let φ_+ and φ_- be two positive linear forms on A such that $\varphi = \varphi_+ - \varphi_-$. In order that $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$, it is necessary and sufficient that, for an arbitrary positive number ε, there exists a positive element a of A such that $\|a\| \leq 1$, $\|\varphi_+\| - \varphi_+(a) < \varepsilon$ and $\varphi_-(a) < \varepsilon$. If A is a von Neumann algebra, then we can choose a projection as a. Furthermore, for such φ_+, φ_- and a satisfying the conditions, we have*

$$\|\varphi_+ - a\varphi\| \leq \sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}, \quad \|\varphi_- + (1-a)\varphi\| \leq \sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2},$$
$$\|\varphi_+ - a\varphi a\| \leq 2\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}, \quad \|\varphi_- + (1-a)\varphi(1-a)\| \leq 2\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}.$$

Conversely, if the above first and third inequalities are satisfied, then we have

$$\|\varphi_+\| - \varphi_+(a) \leq 3\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2} \quad \varphi_-(a) \leq 4\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}.$$

At last, such positive linear forms φ_+ and φ_- are unique.

Proof. When A is not unital, adjoin the identity 1 and extend φ_+ and φ_- to positive linear forms. Suppose that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$ for two positive linear forms φ_+ and φ_- on A. For an arbitrary positive number ε, there exists some self-adjoint element x of A such that $\|x\| \leq 1$ and $\|\varphi\| - \varepsilon < \varphi(x)$. We have

$$\varphi(x) = (\varphi_+(x_+) + \varphi_-(x_-)) - (\varphi_+(x_-) + \varphi_-(x_+)) \leq \varphi_+(x_+) + \varphi_-(x_-).$$

Since $x_+ + x_- \leq 1$, we have

$$\|\varphi_+\| + \|\varphi_-\| - \varepsilon < \varphi_+(x_+) + \varphi_-(1 - x_+),$$

i.e., $\varphi_+(1-x_+)+\varphi_-(x_+)<\varepsilon$. Hence we have $\varphi_+(1-x_+)<\varepsilon$ and $\varphi_-(x_+)<\varepsilon$. If A is a W^*-algebra, then we have $x_+ \le s(x_+)$ and $x_- \le 1-s(x_+)$, so that $\varphi_+(1-s(x_+))<\varepsilon$ and $\varphi_-(s(x_+))<\varepsilon$. Put $a=x_+$ or $a=s(x_+)$.

Conversely, suppose that, for an arbitrary positive number ε, there exists a positive element a of A such that $||a|| \le 1$, $\varphi_+(1-a)<\varepsilon$ and $\varphi_-(a)<\varepsilon$; then we have

$$||\varphi_+||+||\varphi_-|| = \varphi_+(1)+\varphi_-(1) \le \varphi(a-(1-a))+4\varepsilon.$$

Since $||a-(1-a)|| \le 1$, we obtain $||\varphi_+||+||\varphi_-|| \le ||\varphi|| \le ||\varphi_+||+||\varphi_-||$.

If $\varphi_+(1-a)<\varepsilon$ and $\varphi_-(a)<\varepsilon$ for a positive element with $||a|| \le 1$, then we have, for any element x of A,

$$\begin{aligned}|\varphi_+(x)-a\varphi(x)| &\le |\varphi_+(x(1-a))|+|\varphi_-(xa)| \\ &\le \varphi_+((1-a)^2)^{1/2}\varphi_+(xx^*)^{1/2}+\varphi_-(a^2)^{1/2}\varphi_(xx^*)^{1/2} \\ &\le \varphi_+(1-a)^{1/2}||\varphi_+||^{1/2}||x||+\varphi_-(a)^{1/2}||\varphi_-||^{1/2}||x|| \\ &\le \sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}||x||.\end{aligned}$$

We have

$$\begin{aligned}|a\varphi(x)-a\varphi a(x)| = |\varphi((1-a)xa)| &\le |\varphi_+((1-a)xa))|+|\varphi_-((1-a)xa))| \\ &\le \varphi_+(axx^*a)^{1/2}\varphi_+((1-a)^2)^{1/2}+\varphi_-(a^2)^{1/2}\varphi_-((1-a)xx^*(1-a))^{1/2} \\ &\le \sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}||x||,\end{aligned}$$

so that $||\varphi_+ - a\varphi a|| \le 2\sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}$. Similarly, we have

$$||\varphi_- + (1-a)\varphi|| \le \sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2} \quad \text{and} \quad ||\varphi_- + (1-a)\varphi(1-a)|| \le 2\sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}.$$

Conversely, if the above first and third inequalities are satisfied, then we have

$$|\varphi_+(1)-\varphi(a)| \le \sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}, \qquad |\varphi_+(a)-\varphi(a^2)| \le \sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2},$$
$$|\varphi_+(1)-\varphi(a^2)| \le 2\sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}.$$

Hence we have $|\varphi_+(1)-\varphi_+(a)| \le 3\sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}$ and $\varphi_-(a) \le 4\sqrt{2}\varepsilon^{1/2}||\varphi||^{1/2}$.

Let $\varphi_+, \varphi_-, \varphi'_+$ and φ'_- be positive linear forms on A such that

$$\varphi = \varphi_+ - \varphi_- = \varphi'_+ - \varphi'_-$$
$$||\varphi|| = ||\varphi_+||+||\varphi_-|| = ||\varphi'_+||+||\varphi'_-||.$$

For an arbitrary positive number ε, there exists a positive element a of A such that $||a|| \le 1$, $\varphi_+(1-a)<\varepsilon$ and $\varphi_-(a)<\varepsilon$. We have

$$\varphi'_+(a) \ge \varphi'_+(a)-\varphi'_-(a) = \varphi_+(a)-\varphi_-(a) > \varphi_+(1)-2\varepsilon.$$

Similarly we have $\varphi'_-(1-a) > \varphi_-(1) - 2\varepsilon$ and so

$$\varphi'_+(a) + \varphi'_-(1-a) > \|\varphi\| - 4\varepsilon = \varphi'_+(1) + \varphi'_-(1) - 4\varepsilon.$$

Hence we obtain $\varphi'_+(1-a) < 4\varepsilon$ and $\varphi'_-(a) < 4\varepsilon$. Therefore we have $\|\varphi'_+ - a\varphi\| \leq 2\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}$ and $\|\varphi'_- + (1-a)\varphi\| \leq 2\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}$. On the other hand, we have $\|\varphi_+ - a\varphi\| \leq \sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}$ and $\|\varphi_- + (1-a)\varphi\| \leq \sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}$. Hence we have $\|\varphi_+ - \varphi'_+\| \leq 3\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}$ and $\|\varphi_- - \varphi'_-\| \leq 3\sqrt{2}\varepsilon^{1/2}\|\varphi\|^{1/2}$. Consequently, we obtain $\varphi_+ = \varphi'_+$ and $\varphi_- = \varphi'_-$. □

If $\mathscr{M}$ is a von Neumann algebra, then it is obvious that the polar $(\mathscr{M}_*)^\circ$ in $\mathscr{M}^{**}$ is a σ-weakly closed two-sided ideal of $\mathscr{M}^{**}$. Hence there exists a central projection z_0 of $\mathscr{M}^{**}$ such that $(\mathscr{M}_*)^\circ = (1-z_0)\mathscr{M}^{**}$. Then we have $\mathscr{M}_* = z_0\mathscr{M}^*$. An element of $(1-z_0)\mathscr{M}^*$ is said to be singular. We call z_0 the support projection of $\mathscr{M}_*$.

Corollary 2. *The positive part and negative part of a bounded self-adjoint singular (resp., σ-weakly continuous) linear form on a von Neumann algebra are singular (resp., σ-weakly continuous).*

Proof. Let φ be a bounded self-adjoint linear form on a von Neumann algebra $\mathscr{M}$ and z_0 the support projection of $\mathscr{M}_*$. For an arbitrary positive number ε, there exists a projection $e \in \mathscr{M}$ such that $\|\varphi_+ - e\varphi\| < \varepsilon$, in virtue of Lemma 1. If φ is singular, then is also $e\varphi$. Since $(1-z_0)\mathscr{M}^*$ is closed in $\mathscr{M}^*$, φ_+ belongs to $(1-z_0)\mathscr{M}^*$ and does also φ_-.

If φ is σ-weakly continuous, then we have $e\varphi \in \mathscr{M}_*$. Since $\mathscr{M}_*$ is closed in $\mathscr{M}^*$, φ_+ belongs to $\mathscr{M}_*$ and does also φ_-. □

Proposition 3. *Let $\mathscr{M}$ be a von Neumann algebra and φ a positive linear form on $\mathscr{M}$. Then φ is singular if and only if, for any nonzero projection e in $\mathscr{M}$, there exists a nonzero projection $f \in \mathscr{M}$ such that $f \leq e$ and $\varphi(f) = 0$. If ψ is singular, then, for any nonzero projection e in $\mathscr{M}$, there exists a nonzero projection $f \in \mathscr{M}$ such that $f \leq e$ and $\psi(f) = 0$.*

Proof. Let z_0 denote the support projection of $\mathscr{M}_*$. If φ is not singular, then we have $0 \neq z_0\varphi \in \mathscr{M}_*$. Hence we have $0 \neq s(z_0\varphi) \in \mathscr{M}$. There are no nonzero projections f such that $f \leq s(z_0\varphi)$ and $z_0\varphi(f) = 0$.

Suppose that φ is singular. Let e be a nonzero projection in $\mathscr{M}$; then there is a positive linear form $\psi \in \mathscr{M}_*$ such that $\psi(e) \neq 0$. Put $\omega = e(\psi - \varphi)e$. Since $\omega(z_0 - (1-z_0)) = \|e\psi e\| + \|e\varphi e\|$, we have $\|\omega\| = \|e\psi e\| + \|e\varphi e\|$, so that $e\psi e = \omega_+$ and $e\varphi e = \omega_-$, in virtue of Lemma 1. By the same lemma, for a positive number ε with $\varepsilon < \psi(e)$, there exists a decreasing sequence $(e_n)_n$ of projections in $\mathscr{M}$ such that $(e_n\omega e_n)_+(e_n - e_{n+1}) < 2^{-n-1}\varepsilon$ and $(e_n\omega e_n)_-(e_{n+1}) < 2^{-n-1}\varepsilon$, provided $e_0 = e$. Since $(e_n\omega e_n)_+ = e_n\psi e_n$ and $(e_n\omega e_n)_- = e_n\varphi e_n$, we have $\psi(e_n - e_{n+1}) < 2^{-n-1}\varepsilon$ and $\varphi(e_{n+1}) < 2^{-n-1}\varepsilon$. Hence we have $\varphi(\lim_n e_n) \leq$

$\lim_n \varphi(e_n) = 0$. Since $\psi(e) - \psi(e_n) < \varepsilon$, we obtain $\psi(\lim_n e_n) = \lim_n \psi(e_n) \geq \psi(e) - \varepsilon > 0$ and so $\lim_n e_n \neq 0$.

If ψ is singular, then, by Corollary 2, there is a singular positive linear form ω such that $|\psi(x)| \leq \omega(x)$ for every positive element x of $\mathscr{M}$. Hence we see the last statement. $\square$

φ is singular if and only if $[\varphi]$ is singular.

Theorem 4. *A bounded linear form on a von Neumann algebra is σ-weakly continuous if and only if it is completely additive.*

Proof. Clearly, a σ-weakly continuous linear form is completely additive. Let φ be a bounded completely additive linear form on a von Neumann algebra $\mathscr{M}$. By Proposition 3, for any projection $e \in \mathscr{M}$, there exists a mutually orthogonal family $(e_\iota)_\iota$ of nonzero projections in $\mathscr{M}$ such that $e = \sum_\iota e_\iota$ and $(1 - z_0)\varphi(e_\iota) = 0$. Since $(1 - z_0)\varphi$ is completely additive, we have $(1 - z_0)\varphi(e) = \sum_\iota (1 - z_0)\varphi(e_\iota) = 0$. Therefore we obtain $(1 - z_0)\varphi = 0$, i.e., $\varphi \in \mathscr{M}_*$. $\square$

Other proofs of Theorem 4. (I) Without using the concept of singularity and based on only Jordan decomposition, we can prove Theorem 4. Let φ be a bounded completely additive linear form on $\mathscr{M}$; then we may assume that φ is self-adjoint, because that φ^* is completely additive. Let e be a nonzero projection in $\mathscr{M}$ such that $\varphi(e) > 0$. By Lemma 1, there exists a decreasing sequence $(e_n)_n$ of projections in $\mathscr{M}$ such that $\|(e_n\varphi e_n)_+ - e_{n+1}(e_n\varphi e_n)e_{n+1}\| < 2^{-n-2}\varphi(e)$ and $e_n \leq e = e_0$. It follows that

$$\begin{aligned}\varphi(e_n) - \varphi(e_{n+1}) &= (e_n\varphi e_n)(e_n) - (e_{n+1}\varphi e_{n+1})(e_n)\\ &\leq (e_n\varphi e_n)_+(e_n) - (e_{n+1}\varphi e_{n+1})(e_n) \leq 2^{-n-2}\varphi(e).\end{aligned}$$

Hence we have $\varphi(e_n) \geq 2^{-1}\varphi(e) > 0$. Put $f_0 = \inf_n e_n$. Then, for any positive element $x \in \mathscr{M}$, we have $\varphi(f_0 x f_0) = \lim_n (e_n\varphi e_n)_+(f_0 x f_0) \geq 0$, because of $e_n\varphi e_n(f_0 x f_0) = \varphi(f_0 x f_0)$. Hence we have $f_0\varphi f_0 \geq 0$. Since φ is completely additive, we have $\varphi(f_0) = \lim_n \varphi(e_n) > 0$ and so $f_0 \neq 0$.

Let ψ be a positive linear form in $\mathscr{M}_*$ such that $\psi(f_0) > \varphi_+(f_0)$. Put $\omega = f_0(\psi - \varphi_+)f_0$. By the same discussion as above, there exists a decreasing sequence $(f_n)_n$ of projections in $\mathscr{M}$ such that $f_n \leq f_0$, $\liminf_n \omega(f_n) > 0$ and $f\omega f \geq 0$, where $f = \inf_n f_n$. We have $\liminf_n \omega(f_n) \leq \liminf_n \psi(f_n) = \psi(f)$. Therefore we have $f \neq 0$. Since $0 \leq f\varphi_+ f \leq f\psi f$, $f\varphi_+ f$ is σ-strongly continuous. Since $|\varphi_+(xf)| \leq \varphi_+(1)^{1/2}\varphi_+(fx^*xf)^{1/2}$, $f\varphi_+$ is σ-strongly continuous and so σ-weakly continuous. Since $0 \leq f\varphi_- f \leq f\varphi_+ f$, we have $f\varphi_- \in \mathscr{M}_*$. Hence we obtain $f\varphi \in \mathscr{M}_*$.

There exists a maximal mutually orthogonal family $(f_\iota)_{\iota\in I}$ of nonzero projections in $\mathscr{M}$ such that $f_\iota\varphi \in \mathscr{M}_*$. By the maximality, we have $\varphi(e) = 0$ for every projection $e \in \mathscr{M}$ such that $e \leq 1 - \sum_{\iota\in I} f_\iota$. Putting $e = 1 - \sum_{\iota\in I} f_\iota$, we have $e\varphi e = 0$. If $e \neq 0$, then, by the same discussion as above, there exists a nonzero projection $f \in \mathscr{M}$ such that $f \leq e$ and $f\varphi \in \mathscr{M}_*$, which contradicts the maximality of $(f_\iota)_{\iota\in I}$. Therefore we have $e = 0$, i.e., $\sum_{\iota\in I} f_\iota = 1$. If p denotes the sum of $(f_\iota)_{\iota\in I}$ in $\mathscr{M}^{**}$, then we have $\sum_{\iota\in I}(f_\iota\varphi) = p\varphi \in \mathscr{M}^*$ in the $\sigma(\mathscr{M}^*, \mathscr{M}^{**})$-topology. Since $\mathscr{M}_*$ is $\sigma(\mathscr{M}^*, \mathscr{M}^{**})$-closed in $\mathscr{M}^*$, $p\varphi$ is in $\mathscr{M}_*$. By the same discussion as above, for any nonzero projection $e \in \mathscr{M}$, there exists a mutually orthogonal family $(e_\kappa)_\kappa$ of projections in $\mathscr{M}$ such that $\sum_\kappa e_\kappa = e$ and $e_\kappa\varphi \in \mathscr{M}_*$. It follows that

$$\begin{aligned}\varphi(e_\kappa) = e_\kappa\varphi(1) = e_\kappa\varphi\left(\sum_{\iota\in I} f_\iota\right) &= \sum_{\iota\in I} e_\kappa\varphi(f_\iota) = \sum_{\iota\in I}\varphi(f_\iota e_\kappa)\\ &= \sum_{\iota\in I}\overline{\varphi(e_\kappa f_\iota)} = \sum_{\iota\in I}\overline{f_\iota\varphi(e_\kappa)} = \overline{p\varphi(e_\kappa)}.\end{aligned}$$

Since $\varphi(e_\kappa) \in \mathbf{R}$, we have $p\varphi(e_\kappa) = \overline{\overline{p\varphi(e_\kappa)}} = \overline{\varphi(e_\kappa)} = \varphi(e_\kappa)$. Therefore we have $\varphi(e) = \sum_\kappa \varphi(e_\kappa) = \sum_\kappa p\varphi(e_\kappa) = p\varphi(e)$. Consequently, we obtain $\varphi = p\varphi \in \mathscr{M}_*$.

(II) In the manner of Dixmier, we can prove the theorem. Let φ be a bounded completely additive and self-adjoint linear form on $\mathscr{M}$. Let e be a projection in $\mathscr{M}$ such that $\varphi(e) > 0$; then there exists a maximal mutually orthogonal family (e_ι) of nonzero projections in $\mathscr{M}$ such that $e_\iota \leq e$ and $\varphi(e_\iota) \leq 0$ for every ι. Since φ is completely additive, we have $\varphi(\sum_\iota e_\iota) = \sum_\iota \varphi(e_\iota) \leq 0$ and so $\sum_\iota e_\iota \neq e$. Putting $f_0 = e - \sum_\iota e_\iota \neq 0$, by the maximality, we have $\varphi(f) \geq 0$ for all projections f with $f \leq f_0$ and so $f_0\varphi f_0 \geq 0$.

Let ψ be a positive linear form in $\mathscr{M}_*$ such that $\psi(f_0) > \varphi_+(f_0)$ and put $\omega = \psi - \varphi_+$. Then there exists a maximal mutually orthogonal family $(f_\kappa)_{\kappa\in K}$ of nonzero projections in $\mathscr{M}$ such that $f_\kappa \leq f_0$ and $\omega(f_\kappa) \leq 0$ for every κ. For any finite subset F of K, we have $\sum_{\kappa\in F}\varphi_+(f_\kappa) = \varphi_+(\sum_{\kappa\in F} f_\kappa) \leq \varphi_+(f_0)$. Hence it follows that

$$\psi\left(\sum_{\kappa\in K} f_\kappa\right) = \sum_{\kappa\in K}\psi(f_\kappa) \leq \sum_{\kappa\in K}\varphi_+(f_\kappa) \leq \varphi_+(f_0).$$

Therefore we have $\sum_{\kappa\in K} f_\kappa \neq f_0$. Putting $f = f_0 - \sum_{\kappa\in K} f_\kappa \neq 0$, we have $f\omega f \geq 0$. By the discussion in (I), we have $f\varphi \in \mathscr{M}_*$ and so $\varphi \in \mathscr{M}_*$. □

The above proof shows that any positive linear form on a von Neumann algebra is locally σ-weakly continuous, or equivalently any singular positive linear form on a von Neumann algebra is locally trivial. Generally we give the following proposition.

Proposition 5. *Let φ be a bounded linear form on a von Neumann algebra $\mathscr{M}$ and e a nonzero projection in $\mathscr{M}$. Then there exists a nonzero projection f in $\mathscr{M}$ majorized by e such that φ is σ-weakly continuous on $f\mathscr{M}f$.*

Proof. At first, suppose that φ is self-adjoint; then we have $|\varphi(x_+)| \leq |\varphi|(x_+)$ and $|\varphi(x_-)| \leq |\varphi|(x_-)$. Hence we have $|\varphi(x)| \leq |\varphi|(x_+ + x_-) \leq |\varphi|(1)^{1/2}|\varphi|(x^*x)^{1/2}$. There exists a nonzero projection f majorized by e such that $|\varphi|$ is σ-weakly continuous on $f\mathscr{M}f$. Therefore φ is σ-strongly continuous on $f\mathscr{M}f$.

Secondly, suppose $\varphi \in \mathscr{M}^*$; then there exists a nonzero projection f_1 majorized by e such that $\varphi+\varphi^*$ is σ-weakly continuous on $f_1\mathscr{M}f_1$. Hence there exists a nonzero projection f majorized by f_1 such that $i(\varphi-\varphi^*)$ is σ-weakly continuous on $f\mathscr{M}f$. Therefore φ is σ-weakly continuous on $f\mathscr{M}f$. □

Lemma 6. *For a bounded central linear form τ on a von Neumann algebra $\mathscr{M}$, we have $\|\tau\| = \|\tau|\mathscr{Z}\|$, where $\mathscr{Z}$ denotes the center of $\mathscr{M}$.*

Proof. For any element x of $\mathscr{M}$, there exists an element y of the intersection of $\mathscr{Z}$ and the uniform closure of the convex hull of $\{uxu^* \mid u \text{ is a unitary}\}$. Therefore it follows that

$$\tau(y) \in \overline{\tau(\mathrm{co}(\{uxu^* \mid u \text{ is a unitary}\})} = \{\tau(x)\},$$

so that $\tau(x) = \tau(y)$. Hence we have $|\tau(x)| \leq \|\tau|\mathscr{Z}\|\|y\| \leq \|\tau|\mathscr{Z}\|\|x\|$ and so $\|\tau\| \leq \|\tau|\mathscr{Z}\|$. Therefore we obtain $\|\tau\| = \|\tau|\mathscr{Z}\|$. □

Proposition 7. *A bounded central linear form τ on a von Neumann algebra $\mathscr{M}$ is σ-weakly continuous if its restriction to the center $\mathscr{Z}$ of $\mathscr{M}$ is σ-weakly continuous.*

Proof. At first, let τ be a nonzero trace. Then the support z_0 of the restriction of τ to the center is a nonzero finite projection. If, for, it is infinite, then there exists two projections p and q such that $z_0 = p+q$ and $z_0 \sim p \sim q$. Since $\tau(z_0) = \tau(p)+\tau(q) = 2\tau(z_0)$, we have $\tau(z_0) = 0$ and so $z_0 = 0$. By Proposition 5, there exists a nonzero projection e majorized by z_0 such that τ is σ-weakly continuous on $e\mathscr{M}e$. There exists a maximal mutually orthogonal family $(e_\iota)_{\iota\in I}$ of projections equivalent to e. Since z_0 is finite, I is finite. Put $e_0 = 1-\sum_{\iota\in I} e_\iota$; then, by the Comparability Theorem, there exists a central projection z such that $ze_0 \precsim ze$ and $(1-z)e_0 \succsim (1-z)e$. If $z=0$, then we have $e_0 \succsim e$ which contradicts the maximality of (e_ι). Hence we have $z \neq 0$. If v is a partial isometry such that $e_\iota = vv^*$ and $e = v^*v$, then we have $\tau(xe_\iota) = \tau(v^*xv)$. Since $v^*xv \in e\mathscr{M}e$, $e_\iota\tau$ is σ-weakly continuous. Hence $(\sum_{\iota\in I} e_\iota)\tau$ is σ-weakly continuous. Similarly, $(ze_0)\tau$ is σ-weakly continuous and so $z\tau$ is σ-weakly continuous. Therefore there exists a mutually orthogonal family $(z_\kappa)_{\kappa\in K}$ of nonzero central projections such that $z_\kappa\tau$ is σ-weakly continuous and $\sum_{\kappa\in K} z_\kappa = z_0$. It follows that, for any finite subset F of K

$$\left|\left(z_0-\sum_{\kappa\in F} z_\kappa\right)\tau(x)\right| \leq \tau\left(z_0-\sum_{\kappa\in F} z_\kappa\right)^{1/2}\tau(x^*x)^{1/2} \leq \tau\left(z_0-\sum_{\kappa\in F} z_\kappa\right)^{1/2}\|\tau\|^{1/2}\|x\|.$$

Since $\lim_F \tau(z_0-\sum_{\kappa\in F} z_\kappa) = 0$, $z_0\tau$ is σ-weakly continuous and is also τ.

Secondly, we assume that τ is self-adjoint. Let z denote the support of $(\tau|\mathscr{Z})_+$; then we have $(\tau|\mathscr{Z})_+ = z\tau|\mathscr{Z}$. By Lemma 6, since $z\tau$ is central, we have $\|z\tau\| = \|z\tau|\mathscr{Z}\| = z\tau(1)$

and so $z\tau$ is positive. Since $z\tau$ is σ-weakly continuous on $\mathscr{L}$, it is σ-weakly continuous. Similarly a trace $-(1-z)\tau$ is σ-weakly continuous. Since $\tau = z\tau + (1-z)\tau$, τ is σ-weakly continuous.

For a bounded central linear form τ, $\tau + \tau^*$ and $i(\tau - \tau^*)$ is central and so is σ-weakly continuous. Therefore τ is σ-weakly continuous. $\square$